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The inviscid instability of a Blasius boundary layer at large values of the Mach number

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The unstable and neutral modes of a compressible boundary-layer flow past an insulated flat plate are discussed in the limit of infinite Mach number. These modes have been documented by Mack and many of the asymptotic results derived here are becoming evident in his computations at finite values of the Mach number. Of particular interest is the existence of a vorticity mode for which the wavenumber is a discontinuous function of Mach number at finite Mach number but is continuous in the limit $M_\infty \to \infty$. At large Mach number this is the most unstable mode, and is expected to have relevance also in the hypersonic limit when the flow field is no longer shock-free.

1. Introduction

It is well known that compressible boundary layers are unstable to both inviscid and viscous perturbations. An excellent review of the linear inviscid modes of instability is that of Mack (1987), and in a later paper, Mack (1989) extends his theory and computations to examine the inviscid instability of supersonic shear flows. The viscous theory is also documented (see Mack 1987 and his earlier comprehensive article, 1984). The present study is entirely on inviscid and two-dimensional perturbations, and details the structure of the Mack modes for a Blasius boundary layer as the Mach number increases, for a Chapman fluid. One reason for the investigation is the renewed interest in hypersonic flows as a result of the current development of supersonic and hypersonic transport.

A recent study of the viscous instability of a laminar boundary layer at high Mach number may conveniently be mentioned at this stage. It is due to Smith (1989) who constructed an asymptotic triple-deck description of lower-branch Tollmien–Schlichting waves for three-dimensional disturbances at a sufficient angle to the free-stream direction to be outside the local wave-Mach-cone. In this study a crucial ratio, $M_\infty/R^{\frac{1}{10}}$, where M_∞ is the Mach number of the free stream and R an appropriately defined Reynolds number, was identified. As M_∞ increases so that this ratio becomes of order unity, the development of the Tollmien–Schlichting waves takes place on the same lengthscale as that of the basic flow, and the assumption of quasi-parallelism is no longer justified. This development occurs at a lower value of the Mach number than that, $O(R^{\frac{1}{6}})$, at which the assumptions of a hypersonic boundary layer are generally considered to fail (or change, owing to interaction) and may imply that existing parallel-flow calculations of the Orr–Sommerfeld type are open to question.

The inviscid modes studied here are hypersonic in the sense that the present investigation is the limit of a theory as the free-stream Mach number tends to infinity, but it takes no account of the presence of shocks. The basic flow is that of a shock-free Blasius boundary layer past an insulated flat plate; this is assumed to be a parallel flow. It is doubtful that the shock-free or the parallel-flow assumptions continue to have validity in general for the developing interactive hypersonic boundary-layer flow, described by Stewartson (1964) for example, since the attached leading-edge shock is then a controlling feature, and the shock layer and the boundary layer interact, although it is found that some of the present limiting equations continue to hold within the boundary-layer part even so. The linearized equation for the pressure is given by Mack (1984, 1987) and is

$$\frac{\mathrm{d}^2 p}{\mathrm{d}y^2}-\frac{\mathrm{d}}{\mathrm{d}y}(\log \bar{M}^2)\frac{\mathrm{d}p}{\mathrm{d}y}-\alpha^2(1-\bar{M}^2)\,p = 0, \tag{1.1}$$

with boundary conditions $p'(0) = p(\infty) = 0$. Here α is taken to be real and is the wavenumber in the free-stream direction, $\bar{M}$ is defined as

$$\bar{M} = (u-c)M_\infty/T^{\frac{1}{2}}, \tag{1.2}$$

where u, T are the velocity and temperature profiles of the basic flow, and the eigenvalue c, in general complex, determines the temporal instability of the disturbance.

The various possibilities for the solutions of (1.1) have been comprehensively documented by Mack (1984, 1987) and solutions, both neutral and otherwise, computed for values of M_∞ up to about 10. Lees & Lin (1946) classified the neutral modes as subsonic, for which $1-1/M_\infty < c < 1+1/M_\infty$, sonic, for which $c = 1\pm 1/M_\infty$, and supersonic for which $c < 1-1/M_\infty$. When $\bar{M}^2 < 1$ the disturbance is everywhere relatively subsonic and the theory is similar to that for the incompressible case. However, the situation of present interest has $\bar{M}^2 > 1$ over some portion of the boundary layer, the most interesting case being when the relative supersonic region borders the wall. There is then the opportunity for waves to become trapped between the boundary and the sonic line further out in the boundary layer.

Compressible instability theory permits a multiplicity of solutions that does not occur in the incompressible theory. For the insulated flat plate additional neutral modes appear at $M_\infty \doteqdot 2.2$ when a region of relative supersonic flow first occurs at the wall. Above this Mach number, there are two sequences of neutral modes that are of consequence to this study in that they have neighbouring unstable modes. For the first of these sequences $c = c_s$, where c_s is the value of u at the generalized inflection point, defined in (2.5) below, and for the second $c = 1$. The former sequence features a critical layer which, as $M_\infty \to \infty$, moves out to the edge of the boundary layer, while for the latter the critical layer is at the boundary-layer edge for all M_∞. Those neutral modes which are on a portion of the curve of α against M_∞ that is falling, we term, in agreement with Mack, 'acoustic modes', and those which are on a portion that is rising, we term 'vorticity modes'. They might also be called the minor and the major modes respectively, since the vorticity modes have the greatest growth rates at large Mach numbers. Mack's calculations for the inflectional neutral modes show that $\mathrm{d}\alpha/\mathrm{d}M_\infty < 0$ over the major part of every neutral curve so that the modes there are acoustic, but that each neutral curve has a range of M_∞ for which $\mathrm{d}\alpha/\mathrm{d}M_\infty > 0$. Together these non-overlapping portions of the successive neutral curves form an almost continuous rising curve for all M_∞, and one of the results of the present study is that, as $M_\infty \to \infty$, the segments become the continuous neutral curve of the vorticity mode. Our attention here is directed to the structure of both the acoustic and vorticity neutral modes for $M_\infty \gg 1$, together with examination of the instability of the neighbouring modes, among other features.

The plan of the paper is as follows. The acoustic modes are discussed briefly in §2, a development that parallels, and to a small extent extends, a section of a paper by Cowley & Hall (1990) on the stability of hypersonic flow past a wedge including the effect of the shock. In §3 we examine the vorticity mode as $M_\infty \to \infty$, in which limit it may be regarded as a continuous function of M_∞ (see §4 below). For this mode α^2 is large, specifically $O(\log M_\infty)$, $c \approx c_s$ and the growth rate αc_i is $O(M_\infty^{-2}(\log M_\infty)^{\frac{1}{2}})$. The corresponding acoustic modes have $c \approx c_s$, or $c \approx 1$, depending on whether they are inflectional or non-inflectional, and $\alpha = O(M_\infty^{-2})$. The growth rate is also smaller than for the vorticity mode being $O(M_\infty^{-6}/(\log M_\infty)^{\frac{1}{2}})$, although at low and moderate values of the Mach number it is the so-called second mode, an acoustic mode, that is the most unstable. The growth rate of the vorticity mode, is, to leading order, entirely determined by the equation that holds in the critical layer together with its boundary conditions. Indeed this equation is identical to that obtained by Balsa & Goldstein (1990) in their study of the instabilities of supersonic mixing layers in the hypersonic limit (see also Jackson & Grosch 1989). At large values of the Mach number there is a near-linking of the vorticity mode and the acoustic modes, to within exponentially small regions of the (M_∞, α)-plane. The structure of the linking is demonstrated in §4, the results of which are in qualitative agreement with the calculations of Mack (1984, 1987), and quantitative agreement with those of Cowley & Hall.

A question that remains open here, if something of a side issue, is the precise fate of the vorticity mode as $\alpha^2/\log M_\infty \to 0$. As shown in §3 the proposed structure breaks down in this limit, with the relative phase speed tending to infinity although, to be sure, with the growth rate tending to zero. The scales involved become quite complicated. It was at first suspected that the limit was the neutral sonic mode with $c = 1 - 1/M_\infty$ described by Mack (1987) and shown to be still in existence by his figure 9, for example, at $M_\infty = 7$. However as demonstrated here in §5, this mode is no longer in existence, except perhaps as an isolated neutral mode, for values of M_∞ beyond 75. This cut-off number, for the flat-plate case, is of course an extremely large one, but in general the cut-off value is a profile-dependent quantity and so could take smaller, more realistic, values for certain boundary-layer flows. That could be significant since the cut-off Mach number heralds the onset of new, outward-travelling disturbances persisting at the edge of the boundary layer.

In this study we have used for the basic flow a Blasius boundary-layer flow past an insulated flat plate for a model fluid. This means that the Prandtl number has been taken as unity, instead of the usual 0.72 for air, and the viscosity has been assumed to be proportional to the absolute temperature so that the Chapman constant is one. The effect of these assumptions on the asymptotics at infinite Mach number is likely to be small, but an important consequence is that quantitative comparisons with the numerical results of Mack at finite Mach number may be unproductive. Mack was extremely careful to model the precise dependence of the various quantities on the temperature, and at high temperatures replaced the Chapman viscosity–temperature law with the Sutherland law, and used Keyes formula for the conductivity; for a discussion of this see Mack (1965). His aim was to provide a comparison with experiment and possibly a prediction for transition to turbulence by use of linear stability theory, while ours is to gain some insight into the possible modes of instability of a hypersonic boundary layer. For this reason in the following the simplest base flow with the required properties has been employed. Indeed the inflectional neutral modes of Mack have been recomputed by Cowley & Hall (1990) for unit Prandtl number and the Chapman viscosity temperature law and in §§2 and

4 we are able to make some quantitative comparisons with their results. The present analysis applies also to wakes, and the vorticity mode, for instance, compares well with wake-stability computations kindly supplied by Dr D. T. Papageorgiou as shown in Brown *et al.* (1990).

2. The acoustic modes when $M_\infty \gg 1$

When written in terms of the Howarth–Dorodnitsyn variable

$$z^* = \int_0^y \frac{\mathrm{d}y}{T} \tag{2.1}$$

the compressible Rayleigh equation, (1.1) above, for the pressure becomes

$$\frac{\mathrm{d}^2 p}{\mathrm{d}z^{*2}} - \frac{2\,\mathrm{d}u/\mathrm{d}z^*}{u-c}\frac{\mathrm{d}p}{\mathrm{d}z^*} - \alpha^2 T(T-(u-c)^2 M_\infty^2)\,p = 0, \tag{2.2}$$

with boundary conditions

$$p'(0) = p(\infty) = 0. \tag{2.3}$$

For a unit Prandtl number, and a viscosity that is proportional to the absolute temperature, the temperature T is given by

$$T = 1 + \tfrac{1}{2}(\gamma-1)M_\infty^2(1-u^2) \tag{2.4}$$

in the case of a thermally insulating wall. Here γ is the ratio of the specific heats, and when its numerical value is required this will be taken as 1.4.

As discussed by Mack (1984, 1987), and mentioned here in §1, regular neutral modes of a Blasius boundary layer over an insulated wall occur for three values of c. The first is when $c = 1 - M_\infty^{-1}$ and for this $\alpha = 0$. The second and third occur for two sequences of values of α, with $c = 1$ and $c = c_s$ respectively. Those for which $c = 1$ are the non-inflectional neutral modes, while those for which $c = c_s$ have a generalized inflection point at $z^* = z_s^*$ where

$$T\frac{\mathrm{d}^2 u}{\mathrm{d}z^{*2}} = 2\frac{\mathrm{d}T}{\mathrm{d}z^*}\frac{\mathrm{d}u}{\mathrm{d}z^*} \tag{2.5}$$

with $c_s = u(z_s^*)$. For the Blasius boundary layer for which, when $z^* \gg 1$,

$$u = 1 - \mu z^{-1}\mathrm{e}^{-z^2/4}\{1 + 2z^{-2} + O(z^{-4})\}, \tag{2.6}$$

where $\mu = 0.468$ and $z = z^* - 1.721$, it is easily shown that

$$c_s = 1 - \frac{1}{(\gamma-1)M_\infty^2}\{1 + O(1/\log M_\infty)\}, \tag{2.7}$$

and that the generalized inflection point, or critical layer, is at

$$\tfrac{1}{2}z = \Gamma + O(\Gamma^{-1}) \tag{2.8}$$

where

$$\frac{\mu}{2\Gamma}\mathrm{e}^{-\Gamma^2} = \frac{1}{M_\infty^2}. \tag{2.9}$$

Thus, to leading order

$$\Gamma \approx (\log M_\infty^2)^{\frac{1}{2}}. \tag{2.10}$$

The sequences of eigenvalues α associated with $c = 1$ and $c = c_s$ we denote by α_{1n} and α_{sn} respectively. Since, when $M_\infty \to \infty$, c_s, as given in (2.7), tends to unity, it is not surprising, as is evident in figures 2 and 4 of Mack (1987), that the limiting forms of these modes are the same. It is evident, too, that the neighbouring unstable modes have small growth rate: see Mack's figure 9 for example where in the diagram for $\operatorname{Im} c$ there are little humps of instability above the α-axis, the neutral mode at the left-hand end of the hump having $c = 1$, and that to the right-hand side having $c = c_s$.

These acoustic modes, and neighbouring unstable modes, are discussed in the simultaneous work by Cowley & Hall (1990), so where the work overlaps a very brief account is given, although sufficient of the results are presented, with acknowledgement where appropriate, to lead into the topic of §4. This latter is a discussion of the linking between the present acoustic modes and the vorticity mode of the following section.

When $M_\infty \gg 1$, the leading approximations to α_{1n}, α_{sn} are the same, α_0 say, and are obtained by replacing c by unity and T by $\frac{1}{2}(\gamma-1)M_\infty^2(1-u^2)$ in (2.2) so that

$$\frac{\mathrm{d}^2 p_0}{\mathrm{d}z^{*2}} - \frac{2}{u-1}\frac{\mathrm{d}u}{\mathrm{d}z^*}\frac{\mathrm{d}p_0}{\mathrm{d}z^*} - \beta_0^2(1-u^2)^2\left[1 - \frac{2(1-u)}{(\gamma-1)(1+u)}\right]p_0 = 0, \tag{2.11}$$

subject to $p_0'(0) = p_0(\infty) = 0$. Here $\beta_0 = \frac{1}{2}(\gamma-1)\alpha_0 M_\infty^2$. The first ten values of β_0 are obtained by Cowley & Hall. We have obtained the first value of β_0 independently, and the large α_0 behaviour as described later. To find the correction to β_0 when c is real, or the correction to c when αM_∞^2 is given and the mode is not neutral, we must also consider the critical layer and the potential region outside it, as did Cowley & Hall. We write

$$\tfrac{1}{2}(\gamma-1)M_\infty^2\alpha \equiv \beta = \beta_0 + \beta_1/M_\infty^2, \quad c_1 = 1 - \tilde{c}_1/M_\infty^2, \tag{2.12}$$

and note that if the mode is neutral then $\tilde{c}_1 = 0$ for $\alpha = \alpha_{1n}$, i.e. $\beta = \beta_{1n}$ and $\tilde{c}_1 = 1/(\gamma-1)$ for $\alpha = \alpha_{sn}$, i.e. $\beta = \beta_{sn}$. We shall show that, in agreement with Mack, $\beta_{1n} < \beta_{sn}$, at least for the lowest mode, and shall find the leading approximation to $\operatorname{Im} c$ for $\beta_{1n} < \beta < \beta_{sn}$.

The result we have obtained from the match of the solutions in the regions where $z = O(1)$, the critical layer and the region beyond it is equivalent to that of Cowley & Hall. It is that $\tilde{c}_1$ is determined by the equation

$$\begin{aligned}
&\frac{A_0^2(\gamma-1)\tilde{c}_1^2}{2\mu^4\beta_0}\left\{1 - \frac{2\beta_0\,\mathrm{i}\pi}{(\gamma-1)M_\infty^2\,\Gamma}(1-\tilde{c}_1^2(\gamma-1)^2)\right\} \\
&= \int_0^\infty \frac{p_0}{(1-u)^2}\left\{2\beta_0\beta_1(1-u)^2\left[1-\frac{2(1-u)}{(\gamma-1)(1+u)}\right]p_0\right. \\
&\quad + \frac{4\beta_0^2}{\gamma-1}(1-u^2)\left[1 - \frac{1-u}{(\gamma-1)(1+u)} + \tilde{c}_1(1-u)\right]p_0 \\
&\quad \left. - \frac{2\tilde{c}_1}{(1-u)^2}\frac{\mathrm{d}u}{\mathrm{d}z^*}\frac{\mathrm{d}p_0}{\mathrm{d}z^*}\right\}\mathrm{d}z^*,
\end{aligned} \tag{2.13}$$

where A_0 is a constant depending on the normalization chosen for p_0; here we have taken

$$\frac{\mathrm{d}p_0}{\mathrm{d}z^*} \approx A_0 \frac{\mathrm{e}^{-z^2/2}}{z^2} \tag{2.14}$$

for $z^* \gg 1$. Also, in the derivation of (2.13) we have assumed that $\operatorname{Im}\tilde{c}_1 < 0$.

We see from (2.13) that if $\tilde{c}_1$ is real and the mode is neutral then either $\tilde{c}_1 = 0$ or $\tilde{c}_1 = 1/(\gamma-1)$ and in both cases β_1 follows (say $\tilde{\beta}_1$ and $\tilde{\beta}_s$ respectively). However if β_1 is given, and the work of Mack suggests that $\tilde{\beta}_1 < \beta_1 < \tilde{\beta}_s$, then $\tilde{c}_1$ follows.

We now examine (2.13) a little more closely for the perturbation to the lowest neutral mode to see that the conclusions are consistent with the earlier assumption that $\operatorname{Im}\tilde{c}_1 < 0$, and with the results of Mack at finite M_∞.

For the lowest mode we obtain, on solving (2.11) subject to $p_0'(0) = p_0(\infty) = 0$, that $\beta_0 = 0.680$. Evaluation of the required integrals in (2.13) for the Blasius profile and $\gamma = 1.4$ leads to $\tilde{\beta}_1 = -1.01$ (with $\tilde{c}_1 = 0$) and $\tilde{\beta}_s = 2.32$ (with $\tilde{c}_1 = 1/(\gamma-1)$). To examine the neighbourhoods of $\beta_1 = \tilde{\beta}_1, \tilde{\beta}_s$ more closely, and to calculate $\operatorname{Im}\tilde{c}_1$, we write, in (2.13) $\tilde{c}_1 = \tilde{a} + i\tilde{b}$ and split it into its real and imaginary parts. The result of this is the pair of equations

$$\tilde{a}^2 + \mu_0 \tilde{a} + \nu_0 \beta_1 + \delta_0 = 0, \tag{2.15a}$$

$$\tilde{b}(2\tilde{a} + \mu_0) + \frac{\lambda_0}{M_\infty^2 \Gamma}\tilde{a}^2(1 - (\gamma-1)\tilde{a}^2) = 0, \tag{2.15b}$$

where, for the lowest mode, the calculated values of the constants are

$$\mu_0 = -9.265, \quad \nu_0 = 5.082, \quad \delta_0 = 5.115, \quad \lambda_0 = -10.682. \tag{2.16}$$

We see immediately from (2.15*b*) that $\tilde{b} = O(M_\infty^{-2}\Gamma^{-1})$ so that $\operatorname{Im} c = O(M_\infty^{-4}\Gamma^{-1})$. The procedure now is to obtain $\tilde{a}$ from (2.15*a*) and then $\tilde{b}$ from (2.15*b*), and for consistency the resulting $\tilde{b}$ must be negative. We see from (2.15*b*) that $2\tilde{a}_0 + \mu_0$ cannot vanish (it is non-zero when $\tilde{a} = 0$ or $\tilde{a} = 1/(\gamma-1)$) and therefore remains negative. It follows that $0 < \tilde{a} < 1/(\gamma-1)$ and, from (2.15*a*), since $\partial\beta_1/\partial\tilde{a} > 0$, that for instability β_1 lies between $\tilde{\beta}_1$ and $\tilde{\beta}_s$. This is in agreement with the growth rates in figures 7, 8 and 9 for example of Mack (1987) which show small humps of instability terminated by the non-inflectional neutral modes on the left-hand side and by the inflectional ones on the right-hand side.

When $\alpha_0 \gg 1$ the leading approximation to the neutral modes may be obtained by an application of the WKB method. There are three regions to consider because there is a turning point. We start with the region nearest the wall for which we write (2.2) in the form

$$\frac{d^2p}{dz^{*2}} - \frac{2\,du/dz^*}{u-1}\frac{dp}{dz^*} - \alpha^2 M_\infty^4 G(z^*)\,p = 0, \tag{2.17}$$

where

$$G(z^*) = \tfrac{1}{4}(\gamma-1)^2(1-u^2)^2\left[1 - \frac{2(1-u)}{(\gamma-1)(1+u)}\right]. \tag{2.18}$$

In the neighbourhood of the wall, for $z^* < z_a^*$ say, $G(z^*) < 0$, but has a zero at $z^* = z_a^*$ and is thereafter positive. Here c_s has been replaced by its leading-order term unity for $M_\infty^2 \gg 1$ and $T(T-(u-c)^2M_\infty^2)$ by its form for $M_\infty^2 \gg 1$ and $z^* = O(1)$. The final result (equation (2.25)) will be the leading term for α_0 as a function of M_∞ for both the inflectional and non-inflectional neutral modes.

The solution of (2.17) for $z^* < z_a^*$ that satisfies $p'(0) = 0$ is

$$p(z^*) = E_0 \frac{(1-u)}{(-G(z^*))^{\frac{1}{4}}} \cos\left(\alpha M_\infty^2 \int_0^{z^*} (-G(z_1^*))^{\frac{1}{2}}\, dz_1^*\right), \tag{2.19}$$

where E_0 is a constant and $\alpha M_\infty^2 \gg 1$. In the region containing the turning point the equation to be solved is, as is standard, the Airy equation

$$\frac{d^2p}{dz^{*2}} - \alpha^2 M_\infty^4 \sigma(z^* - z_a^*)\,p = 0, \tag{2.20}$$

n	$(n\pi+\frac{1}{4}\pi)/J_-$	Cowley & Hall
0	1.757	3.401
1	8.785	9.786
2	15.81	16.50
3	22.84	23.36
4	29.87	30.28
5	36.90	37.24
6	43.93	44.23
7	50.95	51.23
8	57.98	58.24
9	65.01	65.26

TABLE 1. Comparison of the eigenvalues αM_∞^2 of the acoustic modes as given by (2.25) with the numerical results of Cowley & Hall (1990)

where $\sigma = G'(z_a^*) > 0$, with the required exponentially decaying solution

$$p = e_0\,[\pi\,\mathrm{Ai}\,((\alpha^2 M_\infty^4\,\sigma)^{\frac{1}{3}}(z^*-z_a^*))]^{\frac{1}{2}}, \tag{2.21}$$

e_0 being an arbitrary constant. If we denote the argument of the Airy function by t^* then, for $t^* \ll -1$,

$$p \approx \frac{e_0}{|t^*|^{\frac{1}{4}}}\cos\left(\tfrac{2}{3}|t^*|^{\frac{3}{2}}-\tfrac{1}{4}\pi\right). \tag{2.22}$$

If we match the amplitude and phase of (2.22) with that of (2.19) as $z^* \rightarrow z_a^*$ we find that

$$\frac{1-u(z_a^*)}{\sigma^{\frac{1}{4}}}(\alpha^2 M_\infty^4\,\sigma)^{\frac{1}{12}}E_0 = \pm e_0, \tag{2.23}$$

and

$$\alpha M_\infty^2 \int_0^{z_a^*} (-G(z^*))^{\frac{1}{2}}\,\mathrm{d}z^* = n\pi+\tfrac{1}{4}\pi, \tag{2.24}$$

where n is any integer as long as it is large. The arguments of §§3 and 4, in which we consider the vorticity mode and its near-linking with these acoustic modes, will require explicit consideration of the region $z^* > z_a^*$ since the vorticity mode is dominant in this region. The principal result of the present section is (2.24) which may be written as

$$\alpha M_\infty^2 = (n\pi+\tfrac{1}{4}\pi)/J_-, \tag{2.25}$$

where

$$J_- = \tfrac{1}{2}(\gamma-1)\int_0^{z_a^*}(1-u^2)\left[\frac{2(1-u)}{(\gamma-1)(1+u)}-1\right]^{\frac{1}{2}}\mathrm{d}z^*. \tag{2.26}$$

With $\gamma = 1.4$, we obtain $J_- = 0.447$, and the values of αM_∞^2 obtained from (2.25) are compared, in table 1, with the eigenvalues given by Cowley & Hall (1990) obtained by direct integration of equation (2.11). Their results have been divided by $\surd 2$ for the comparison as the α of their paper is $\surd 2$ times that considered here.

The formula $(n\pi-\frac{1}{4}\pi)/J_-$ as given by Mack (1987) appears to be in error. The Airy region has not been properly accounted for, and comparison with the results of Cowley & Hall is poor, the predicted eigenvalues lying almost half-way between the computed ones. We see from table 1 that the prediction of the corrected formula is very satisfactory.

3. The vorticity mode when $M_\infty \gg 1$

The computations of Mack at low and moderate values of M_∞ (up to $M_\infty = 10$) for the inflectional neutral modes of a Blasius boundary layer over an insulated wall are of concern here. These modes, as noted in §2, have $c = c_s$, c_s being the value of u at the generalized inflection point. The computations show that, at each value of M_∞, one of the modes has α as an increasing function of M_∞. The range of M_∞ for which this occurs transfers to the successive modes as M_∞ increases; this phenomenon is clearly visible in figure 2 of Mack's 1987 paper. This discontinuous 'mode' resembles the unique low-speed instability wave that occurs at small Mach numbers, and is termed by Lees (1947) the vorticity mode, those neutral modes for which α decreases as M_∞ increases being the acoustic modes.

We now examine this mode as $M_\infty \to \infty$ and show that in this limit it may be regarded as a virtually continuous function of M_∞. It emerges that it is far more unstable than the acoustic modes, even though it is near-neutral in that $c = 1+O(M_\infty^{-2})$. It is concentrated in the neighbourhood of the critical layer which, when $M_\infty \gg 1$, is situated at $z \approx 2\Gamma$ where Γ is defined by (2.9), and exists, as we shall see in (3.17) below, for values of α such that $\alpha = O(\Gamma)$. Since $\alpha M_\infty^2 \gg 1$ and $c \approx 1$ the region away from the critical layer can be discussed by an adaptation of the WKB analysis of §2. Indeed the appropriate equation is again (2.17) and the solution below the turning point at z_a^* is (2.19). However the solution (2.21) of the Airy equation must be replaced by

$$p = e_0 \pi^{\frac{1}{2}} \operatorname{Ai}(t^*) + f_0 \pi^{\frac{1}{2}} \operatorname{Bi}(t^*) \tag{3.1}$$

since we are not now seeking a solution with exponential decay beyond the turning point. Thus, for $t^* \ll -1$,

$$p \approx (e_0^2+f_0^2)^{\frac{1}{2}} |t^*|^{-\frac{1}{4}} \cos\left(\tfrac{2}{3}|t^*|^{\frac{3}{2}} + \tfrac{1}{4}\pi - \phi\right), \tag{3.2}$$

where $\tan\phi = e_0/f_0$. If we match the amplitude and phase of (3.2) with those of (2.19) as $z^* \to z_a^*$ we find that the analogues of (2.23), (2.24) are

$$(1-u(z_a^*))\, \sigma^{-\frac{1}{4}} (\alpha^2 M_\infty^4 \sigma)^{\frac{1}{12}} E_0 = \pm (e_0^2 + f_0^2)^{\frac{1}{2}} \tag{3.3}$$

and

$$\alpha M_\infty^2 \int_0^{z_a^*} (-G(z^*))^{\frac{1}{2}} \, dz^* = n\pi + \phi - \tfrac{1}{4}\pi \tag{3.4}$$

respectively.

When $t^* \gg 1$ we obtain from (3.1) that

$$p \approx f_0 t^{*-\frac{1}{4}} \exp\left(\tfrac{2}{3} t^{*\frac{3}{2}}\right). \tag{3.5}$$

This form for p must be matched to the solution in the region $z^* > z_a^*$ but below the critical layer where $z^* - 1.721 = O(\Gamma)$ as in (2.8). In this region

$$p(z^*) = \frac{1-u}{(G(z^*))^{\frac{1}{4}}} h_0 \exp\left[\alpha M_\infty^2 \int_{z_a^*}^{z^*} (G(z_1^*))^{\frac{1}{2}} \, dz_1^*\right], \tag{3.6}$$

where h_0 is an arbitrary constant. Here we have retained only the exponentially large term in the WKB approximation because there is no formal means of evaluating the coefficient of the exponentially smaller term (however see §4 below). A match between (3.5) and (3.6) yields

$$f_0/h_0 = (1-u(z_a^*))\, \sigma^{-\frac{1}{4}} (\alpha^2 M_\infty^4 \sigma)^{\frac{1}{12}}. \tag{3.7}$$

We now require the behaviour of (3.6) as the critical layer is entered from below. The critical layer has $\bar{z} = O(1)$ where

$$\tfrac{1}{2} z = \Gamma + \bar{z}/2\Gamma \tag{3.8}$$

as suggested by (2.8), and in its neighbourhood the two terms of T in (2.4) are of the same size with

$$T \approx 1+(\gamma-1)\,\mathrm{e}^{-\bar{z}}, \tag{3.9}$$

$$u-c \approx \frac{1}{M_\infty^2}(c_1-\mathrm{e}^{-\bar{z}}), \tag{3.10}$$

where we have taken

$$c = 1-\frac{c_1}{M_\infty^2}+o(M_\infty^{-2}). \tag{3.11}$$

It emerges that, to ensure a match of the $O(1)$ factors as well as of the exponents, it is necessary to retain the contribution of $\alpha^2 M_\infty^4\, G(z^*)$ in (2.18) when $\bar{z}$ as in (3.8) is of order unity. By use of (3.9), (3.10) it follows that

$$M_\infty^4\, G(z^*) \approx \{1+(\gamma-1)\,\mathrm{e}^{-\bar{z}}\}^2, \tag{3.12}$$

whereupon

$$M_\infty^2 \int_{z_a^*}^{z^*} (G(z_1^*))^{\frac{1}{2}}\,\mathrm{d}z_1^* \approx M_\infty^2\, J_+ + \frac{1}{\Gamma}\{\bar{z}+(\gamma-1)(1-\mathrm{e}^{-\bar{z}})\}, \tag{3.13}$$

where

$$J_+ = \tfrac{1}{2}(\gamma-1)\int_{z_a^*}^{\infty}(1-u^2)\left[1-\frac{2(1-u)}{(\gamma-1)(1+u)}\right]^{\frac{1}{2}}\mathrm{d}z^*. \tag{3.14}$$

Thus, from (3.6), just below the critical layer

$$p \approx \frac{h_0}{M_\infty}\mathrm{e}^{-\bar{z}}\{1+(\gamma-1)\,\mathrm{e}^{-\bar{z}}\}^{-\frac{1}{2}}\exp\left[\alpha M_\infty^2\, J_+ + \frac{\alpha}{\Gamma}\{\bar{z}+(\gamma-1)(1-\mathrm{e}^{-\bar{z}})\}\right]. \tag{3.15}$$

Equation (3.15) will provide a boundary condition for the solution in the critical layer which we now consider.

In the region where $\bar{z} = O(1)$ we make the transformation (3.8) in (2.2), define $\bar{\alpha}$ by

$$\alpha = \Gamma\bar{\alpha} \tag{3.16}$$

and retain the leading terms for $M_\infty \gg 1$. This, upon use of (3.9), (3.10) leads to

$$\frac{\mathrm{d}^2 p}{\mathrm{d}\bar{z}^2}-\frac{2\mathrm{e}^{-\bar{z}}}{c_1-\mathrm{e}^{-\bar{z}}}\frac{\mathrm{d}p}{\mathrm{d}\bar{z}}-\bar{\alpha}^2\{1+(\gamma-1)\,\mathrm{e}^{-\bar{z}}\}^2 p = 0. \tag{3.17}$$

This equation is to be solved with $p \to 0$ as $\bar{z} \to \infty$ so that p decays at the outer edge of the boundary layer, and also $p \to 0$ as $\bar{z} \to -\infty$ to match with (3.15). From (3.15) we see that the precise behaviour of p in (3.17) as $\bar{z} \to -\infty$ must be a multiple of

$$s^{\frac{1}{2}-\bar{\alpha}}\,\mathrm{e}^{-\bar{\alpha}s} \tag{3.18}$$

where

$$s = (\gamma-1)\,\mathrm{e}^{-\bar{z}}. \tag{3.19}$$

If we now write

$$p = s^{\bar{\alpha}}L(s), \quad c_1 = C/(\gamma-1), \tag{3.20}$$

in (3.17) so that it becomes

$$(C-s)\,sL''+[(2\bar{\alpha}+1)\,C+(1-2\bar{\alpha})\,s]L'+\bar{\alpha}[2(1-\bar{\alpha}C)+\bar{\alpha}(2-C)\,s+\bar{\alpha}s^2]L = 0 \tag{3.21}$$

it is easy to verify that the solution of (3.21) with $L(\infty) = 0$ leads to the required behaviour (3.18) for p. Thus the appropriate boundary conditions that will determine the eigenvalue C are

$$L(0) = L(\infty) = 0.$$

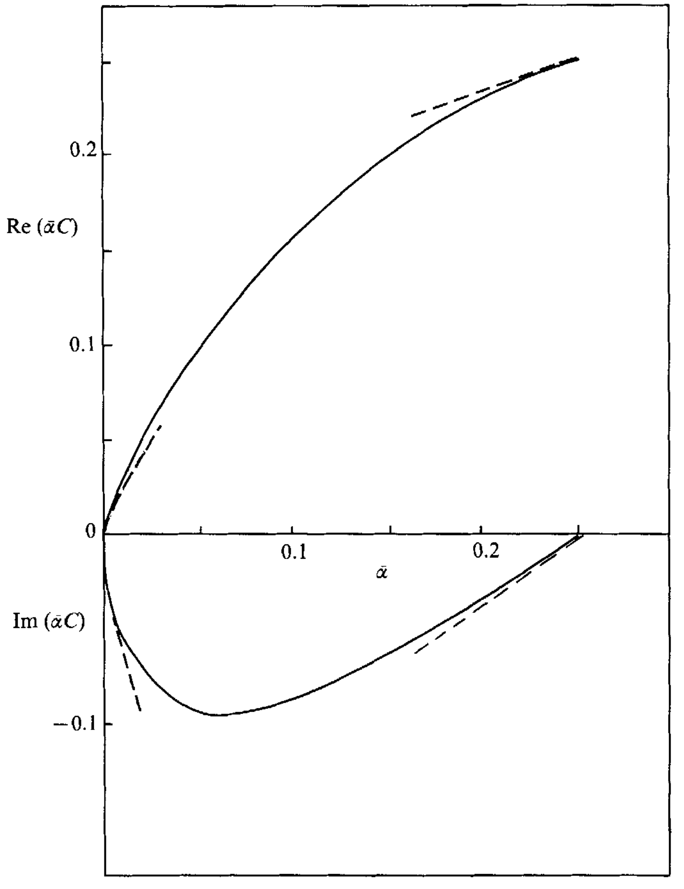

FIGURE 1. The real and imaginary parts of $\bar{\alpha}C$ for $0 \leqslant \bar{\alpha} \leqslant \frac{1}{4}$. The broken curves represent the asymptotic expansions (3.26) as $\bar{\alpha} \to \frac{1}{4}$ and (3.30) as $\bar{\alpha} \to 0$.

Equation (3.17) is, on making the identifications $p = s^{-\frac{1}{2}}(c_1 - s)\mathscr{R}(\zeta)$, $\zeta = e^{-\bar{z}}$, $c_1 = \sigma$, exactly equation (31 *a*) of Balsa & Goldstein (1990) describing the critical-layer region in either of the external streams of a supersonic mixing layer in the hypersonic limit. There, as here, the eigenvalue is, at leading order, determined entirely by the solution in the critical layer. Balsa & Goldstein also obtain numerical solutions of the equation and we therefore proceed with emphasis on the asymptotic forms at limiting values of $\bar{\alpha}$.

In figure 1 we present the real and imaginary parts of $\bar{\alpha}C$ found by numerical solution of (3.21). It is seen that solutions exist for $0 \leqslant \bar{\alpha} \leqslant \frac{1}{4}$, that the mode is neutral at $\bar{\alpha} = \frac{1}{4}$, and that $\bar{\alpha}C \to 0$ as $\bar{\alpha} \to 0$. It is straightforward to examine the behaviour of the eigensolution and of C at the limits of the range of $\bar{\alpha}$ and we do this below.

(i) *The limit* $\bar{\alpha} \to \frac{1}{4}$

When $\bar{\alpha} = \frac{1}{4}$, equation (3.21) has a solution $L = e^{-\frac{1}{4}s}$ with $C = 1$, which is consistent with (2.7). If we perturb $\bar{\alpha}$ from this value and write

$$\bar{\alpha} = \tfrac{1}{4} + \bar{\alpha}_1, \quad C = 1 + C_1, \quad L = e^{-\frac{1}{4}s} + \bar{\alpha}_1 L_1, \tag{3.22}$$

it is found that L_1 satisfies

$$(1-s)\,sL_1'' + \tfrac{1}{2}(3+s)\,L_1' + \tfrac{1}{16}(6+s+s^2)\,L_1 = \tfrac{1}{2}e^{-\frac{1}{4}s}((C_1/\bar{\alpha}_1) - 1 - 2s - s^2). \tag{3.23}$$

Here $C_1/\bar{\alpha}_1$ is to be chosen so that L_1 is finite at the origin, tends to zero at infinity, and is continued analytically through the critical layer at $s = 1$. The result of this analysis, for which it was necessary to evaluate

$$\int_0^\infty (1-s^2)\log|1-s^2|\exp(-\tfrac{1}{2}s^2)\,\mathrm{d}s \tag{3.24}$$

numerically, giving the value -0.68988, is that for consistency $\bar{\alpha}_1 < 0$, and that

$$C_1/\bar{\alpha}_1 = -2.628+2.756\mathrm{i}. \tag{3.25}$$

Thus

$$\bar{\alpha}C = \tfrac{1}{4}-(\tfrac{1}{4}-\bar{\alpha})(0.343+0.689\mathrm{i}), \tag{3.26}$$

which is in good agreement with the results of figure 1.

(ii) *The limit* $\bar{\alpha}\to 0$

This limit is slightly more subtle than that discussed above, since the neighbourhood of $\tfrac{1}{2}z = \Gamma$ that we are examining spreads as $\bar{\alpha}\to 0$, and the critical layer moves towards the wall, although it remains at an asymptotically large distance from it. It is necessary to examine two regions, namely $s = O(\bar{\alpha}^{-1})$ and $s = O(\bar{\alpha}^{-\frac{1}{2}})$, the latter of which contains the critical layer. In the former the solution of (3.21) is, to leading order,

$$L(s) = \bar{\alpha}sK_1(\bar{\alpha}s), \tag{3.27}$$

where K_1 is a Bessel function of the second kind. In the latter region, with $s = (2/\bar{\alpha})^{\frac{1}{2}}t$ and $L = 1+2\bar{\alpha}\bar{L}_1(t)+o(\bar{\alpha})$,

$$\frac{\mathrm{d}\bar{L}_1}{\mathrm{d}t} = \frac{1}{t}[-\tfrac{1}{2}-D_0(t-D_0)+(t-D_0)^2(\log(t-D_0)+A_1)], \tag{3.28}$$

where

$$C = (2/\bar{\alpha})^{\frac{1}{2}}D_0. \tag{3.29}$$

We now choose the constant A_1 to make $\mathrm{d}\bar{L}_1/\mathrm{d}t$ regular at the origin, and match (3.28) as $t\to\infty$ with (3.27) as $\bar{\alpha}s\to 0$. The result is that

$$\mathrm{Re}\,(C\bar{\alpha}) \approx \frac{(2\bar{\alpha})^{\frac{1}{2}}}{(-\log\bar{\alpha})^{\frac{3}{2}}}\tfrac{1}{2}\pi, \tag{3.30a}$$

$$\mathrm{Im}\,(C\bar{\alpha}) \approx -\frac{(2\bar{\alpha})^{\frac{1}{2}}}{(-\log\bar{\alpha})^{\frac{1}{2}}}\left[1-\frac{\log(-\log\bar{\alpha})}{2(-\log\bar{\alpha})}+\frac{d}{(-\log\bar{\alpha})}\right], \tag{3.30b}$$

where

$$d = 1-\tfrac{1}{2}\log 2+\gamma_{\mathrm{e}} \tag{3.31}$$

and $\gamma_{\mathrm{e}} = 0.57722$ is Euler's constant.

The asymptotic forms as $\bar{\alpha}\to\frac{1}{4}$ and $\bar{\alpha}\to 0$ are shown in figure 1 by dashed lines. We note, see also §5 below, that (3.30) implies that the relative phase speed tends to infinity although the growth rate of the disturbance tends to zero as $\bar{\alpha}\to 0$.

We turn now to the correction to the neutral vorticity mode for $M_\infty \gg 1$ of which the relative order is $O(\Gamma^{-2})$ for c and $\bar{\alpha}$. We write

$$c \approx 1-\frac{1}{(\gamma-1)M_\infty^2}+\frac{1}{(\gamma-1)\Gamma^2M_\infty^2} \tag{3.32}$$

and $\bar{\alpha} = \frac{1}{4}+\tilde{\alpha}_1/\Gamma^2$, where the coefficient of the term $O(\Gamma^{-2}M_\infty^{-2})$ may either be written down immediately from (2.5) on use of the asymptotic expansion (2.6) of the Blasius function for $z \gg 1$, or determined during the course of the calculation to ensure that the solution is regular at the critical layer. Thus in (3.32) the given terms are the leading terms in the asymptotic expansion of c_{s} for $M_\infty \gg 1$.

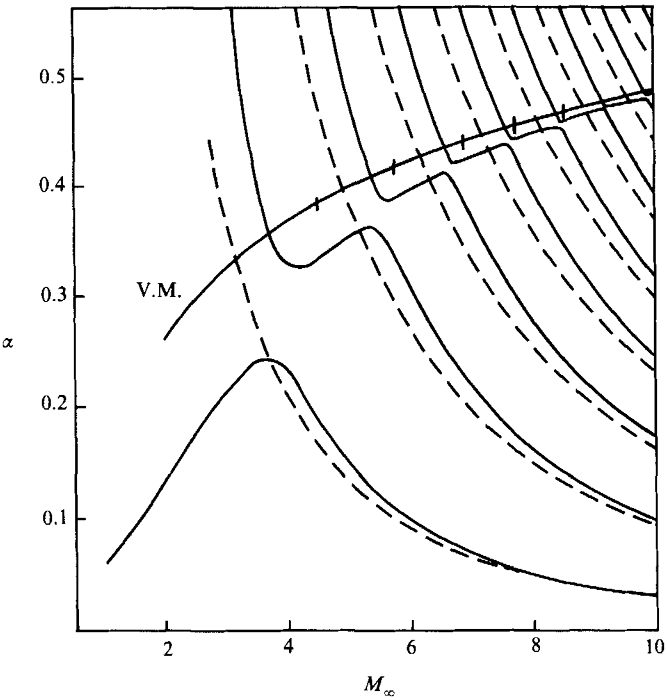

FIGURE 2. The vorticity mode (continuous curve V.M.) superimposed on the numerical calculations (continuous curves) and asymptotic expansion (broken curves) of Cowley & Hall (1990). The vertical marks on V.M. give the values of M_∞ at the near-linkages predicted by (4.9).

The equation to be solved to determine $\tilde{\alpha}_1$ is obtained from (2.2) on retaining the terms in (2.6) which lead to a correction $O(\Gamma^{-2})$ when the transformation (3.19) is made. A correction to (3.20) is made in the form

$$p \approx s^{\frac{1}{4}} e^{-\frac{1}{4}s} + (1/\Gamma^2)\, p_1(s), \tag{3.33}$$

as it is the neutral mode that we are seeking. The result of the calculation, which is tedious but straightforward and in which all the integrals may be evaluated explicitly, is that

$$\tilde{\alpha}_1 = \tfrac{1}{8}(1 + \log \tfrac{1}{2}(\gamma - 1) - \psi(\tfrac{1}{2})), \tag{3.34}$$

i.e. $\tilde{\alpha}_1 = 0.1692$ when $\gamma = 1.4$. In terms of M_∞^2 the contribution from the definition of Γ in (2.9) to α in (3.16) is formally greater than that from $\tilde{\alpha}_1$ in (3.34). From (3.16) and (2.9) we obtain

$$\alpha \approx \tfrac{1}{4}(\log M_\infty^2)^{\frac{1}{2}} \left[1 - \frac{\log(\log M_\infty^2)^{\frac{1}{2}}}{2 \log M_\infty^2} + \frac{\log \frac{1}{2}\mu + 8\tilde{\alpha}_1}{2 \log M_\infty^2} \right], \tag{3.35}$$

with a relative error $O(\Gamma^{-2} \log \Gamma)$. The values of α calculated from (3.35) are superimposed on figure 6 of Cowley & Hall (1990) as the continuous curve labelled V.M. in figure 2 here. Figure 6 of Cowley & Hall is a recomputation of figure 2 of Mack (1987) for unit Prandtl number and Chapman viscosity law, rather than for a Prandtl number of 0.72 and the Sutherland law employed by Mack. In this figure of Cowley & Hall we have recalibrated the ordinate because, as noted in §2, the α of that paper is $\surd 2$ times that considered here. The continuous curves shown (other than V.M.) are their inflectional neutral modes, and the dashed curves the leading term of the asymptotic expansion found by them for the acoustic modes. Further reference to this is made in §4. It is seen that V.M. is an extremely good approximation to the vorticity mode by the time M_∞ is 10. If the wall is not thermally insulating but is at a given temperature, T_w say, then (2.4) is replaced by

$$T = 1+\tfrac{1}{2}(\gamma-1)M_\infty^2(u-u^2)+(T_w-1)(1-u). \tag{3.36}$$

If $T_w = o(M_\infty^2)$ the vorticity mode is the same as described here, but with $\gamma-1$ replaced by $\frac{1}{2}(\gamma-1)$ (since, from (3.36), for $z^* \gg 1$, $T \approx 1+\frac{1}{2}(\gamma-1)M_\infty^2(1-u)$ for large Mach number). However, at finite Mach number, computations analogous to those of Mack will differ.

4. The near-linking of the acoustic modes and the vorticity mode

In this section we analyse the near mode-crossing of the inflectional acoustic neutral modes and the vorticity mode that is evident in figure 2 of Mack's (1987) paper, and in figure 6 of Cowley & Hall (1990); the latter figure is reproduced here as figure 2 with the vorticity mode superimposed thereon. Here the analysis is carried out for $M_\infty \gg 1$ and thus the near-linking occurs for values of αM_∞^2, the appropriate scaled wavenumber for the acoustic modes, that are large, so that $\alpha \approx \frac{1}{4}\Gamma$, the neutral wavenumber of the vorticity mode. The near-linking is, as suggested by Cowley & Hall, of an exponential nature.

To discuss the near-linking we extend the WKB analysis of §§2 and 3. Away from the critical layer and in the region below the turning point z_a^* the solution is again (2.19). In the neighbourhood of z_a^*, p is given by (3.1), and (3.2)–(3.4) also hold. However we replace (3.5) by

$$p \approx t^{*-\frac{1}{4}}(\tfrac{1}{2}e_0 \exp(-\tfrac{2}{3}t^{*\frac{3}{2}})+f_0 \exp(\tfrac{2}{3}t^{*\frac{3}{2}})), \tag{4.1}$$

retaining (with reservations) the exponentially small term. Our justification for this, which is formally correct only if $f_0 \equiv 0$, is that if f_0 is small, which we assume below, the effect of the indeterminate exponentially small terms in the asymptotic expansion of Bi (t^*) will be negligible. We also retain in (3.6) the corresponding term with the negative exponent with an arbitrary multiplier g_0 in addition to the term in h_0. Then, as well as (3.7), we have, from the match with (4.1), that

$$e_0/2g_0 = f_0/h_0. \tag{4.2}$$

The behaviour as the critical layer is entered from below is now required. This is (3.15) augmented by a corresponding term in g_0, namely

$$p \approx \frac{e^{-\bar{z}}}{M_\infty}(1+(\gamma-1)e^{-\bar{z}})^{-\frac{1}{2}}\{g_0 \exp[-\alpha M_\infty^2 J_+ - (\alpha/\Gamma)(\bar{z}+(\gamma-1)(1-e^{-\bar{z}}))]$$
$$+h_0 \exp[\alpha M_\infty^2 J_+ + (\alpha/\Gamma)(\bar{z}+(\gamma-1)(1-e^{-\bar{z}}))]\}. \tag{4.3}$$

In the critical layer we make the transformation (3.8) so that the appropriate equation is again (3.17) with $c_1 = 1/(\gamma-1)$ since it is a neutral mode that we are seeking. In (3.17) we set

$$\bar{\alpha} = \tfrac{1}{4}+\tilde{\alpha}, \quad p = s^{\frac{1}{4}}e^{-\frac{1}{4}s}+\tilde{p}_1, \tag{4.4}$$

where s is defined by (3.19), to give a perturbation to the neutral vorticity mode of §3. If we now write $p = s^{\bar{\alpha}}L$ and $L = e^{-\frac{1}{4}s}+\tilde{L}_1$, then the equation satisfied by $\tilde{L}_1$ is (3.23) but with $C_1 = 0$. On ensuring that the solution is regular at $s = 0$, so that p decays above the critical layer, it is easy to show that, for $s \gg 1$,

$$p \approx s^{\frac{1}{4}}e^{-\frac{1}{4}s} - \tilde{\alpha}(2\pi)^{\frac{1}{2}}s^{\frac{3}{4}}e^{\frac{1}{4}s}, \tag{4.5}$$

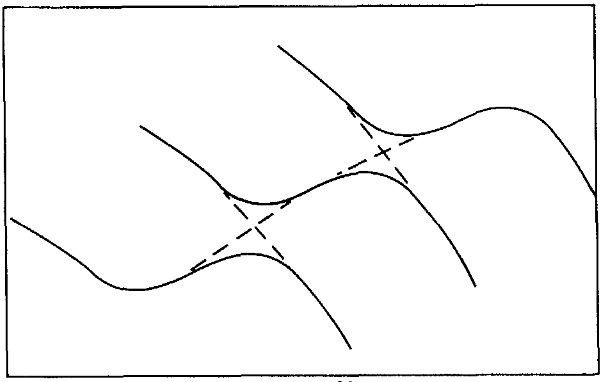

FIGURE 3. Sketch of the near-linking of the acoustic and vorticity modes. The continuous curves represent the finite Mach number configuration and the broken curves the asymptotic limit as $M_\infty \to \infty$.

which implies a matching condition for the solution below the critical layer of

$$p \approx (\gamma-1)^{\frac{1}{4}} \exp[-\tfrac{1}{4}(\gamma-1)\,\mathrm{e}^{-\bar{z}}-\tfrac{1}{4}\bar{z}] - \tilde{\alpha}\surd(2\pi)\,(\gamma-1)^{\frac{3}{4}} \exp[\tfrac{1}{4}(\gamma-1)\,\mathrm{e}^{-\bar{z}}-\tfrac{3}{4}\bar{z}]. \quad (4.6)$$

We now compare (4.3) and (4.6), and see that if we set $\bar{\alpha} \approx \frac{1}{4}$ in (4.3) as in the transformation (4.4), the two terms cross match if

$$\frac{g_0}{h_0} = -\tilde{\alpha}(2\pi)^{\frac{1}{2}}(\gamma-1)^{\frac{1}{2}} \exp[2\alpha M_\infty^2 J_+ + \tfrac{1}{2}(\gamma-1)]. \quad (4.7)$$

We now have g_0/h_0 and hence e_0/f_0 from (4.2). This will give us ϕ in (3.4) which is the crucial result of this section. We assume, as mentioned above, that $\phi \approx \frac{1}{2}\pi - f_0/e_0$, i.e. that $f_0/e_0 \ll 1$, since we are seeking a perturbation not only to the vorticity mode but also to the acoustic modes for which f_0 is zero. The final result is, from (4.7) on use of (4.2), that (3.4) becomes

$$(\alpha - \tfrac{1}{4}\Gamma)\left(\alpha - \frac{n\pi + \frac{1}{4}\pi}{M_\infty^2 J_-}\right) = \frac{\Gamma}{2(2\pi)^{\frac{1}{2}}(\gamma-1)^{\frac{1}{4}}} \frac{\exp[-2\alpha M_\infty^2 J_+ - \frac{1}{2}(\gamma-1)]}{M_\infty^2 J_-}, \quad (4.8)$$

where J_- is defined in (2.26) and J_+ in (3.14).

We are now able to see from (4.8) that, for $M_\infty^2 \gg 1$, the vorticity and acoustic neutral modes are separated by an exponentially small amount. The vanishing of the two factors on the left-hand side leads respectively to the vorticity mode and the acoustic modes. The hyperbolic form of (4.8) for α as a function of M_∞ can be seen in figure 2 of Mack (1987) even at values of M_∞ less than 10. A sketch is given here in figure 3 for a pair of such intersections. The continuous lines represent the situation that occurs at finite M_∞ when the separation of the modes is finite. The dashed lines represent α as given by (4.8) with the exponentially small right-hand side ignored, and lead to the continuous infinite M_∞ limits.

A leading-order approximation for the positions of the near-linking of the successive modes may be made from (4.8). The prediction is

$$\tfrac{1}{4}\Gamma = (n\pi + \tfrac{1}{4}\pi)/M_\infty^2 J_-, \quad (4.9)$$

and the values of M_∞ obtained from this with $\Gamma = (\log M_\infty^2)^{\frac{1}{2}}$ and $n = 1$–5 are shown as vertical marks on the curve V.M. in figure 2. For $n = 6$ and 7 the marks are graphically indistinguishable from the computed near-linkages.

5. The disappearance of the sonic mode at finite M_∞

The results (3.25) of §3 show that the correction c_1 to c in (3.11) is $O(\bar{\alpha}^{-\frac{1}{2}})$ as $\bar{\alpha} \rightarrow 0$. It is therefore necessary to consider a further region in which $\alpha = o(\Gamma)$ in order to determine the fate of the vorticity mode for very small values of α. It was at first expected that in the limit the vorticity mode would finally tend to the neutral sonic mode as described by Mack (1987). This has $c = 1-1/M_\infty$ and $\alpha = 0$. For non-zero α it is unstable and as evident in figures 7, 8 and 9 of Mack's paper where it is labelled S_u, it merges, as M_∞ increases, with an adjacent mode to become the most unstable mode at the Mach numbers he considers. However, as we show below, this sonic mode cannot be the limit of the vorticity mode as $\alpha \rightarrow 0$, because, by the time M_∞ is 75, the sonic mode no longer exists as a neighbouring mode to unstable modes.

To demonstrate this it is easier to consider the equation for $V(\equiv v/(u-c))$, as given by Mack (1984) in his equation (9.8), as the matched asymptotic expansions yield the required results at one stage previous to that if equation (2.2) for p is used. In terms of z^* the equation for V is

$$T\frac{\mathrm{d}}{\mathrm{d}z^*}\left(\frac{1}{T}\frac{\mathrm{d}V}{\mathrm{d}z^*}\right)+\frac{\mathrm{d}}{\mathrm{d}z^*}\left(\log\frac{\bar{M}^2}{1-\bar{M}^2}\right)\frac{\mathrm{d}V}{\mathrm{d}z^*}-\alpha^2T^2(1-\bar{M}^2)\,V=0, \tag{5.1}$$

where $\bar{M}$ is defined in (1.2), and $V(0)=0$ with V bounded as $z^* \rightarrow \infty$.

For the sonic mode $\alpha = 0$ and $c = 1-1/M_\infty$ so we now perturb away from $\alpha = 0$ and set

$$c = 1-\frac{1}{M_\infty}+\alpha^2\tilde{c}+o(\alpha^2), \tag{5.2}$$

where $\tilde{c}$ will be complex and is to be found. The value of M_∞ is to be considered as finite and fixed.

If V and $\bar{M}^2/(1-\bar{M}^2)$ are written in the form

$$V = V_0+\alpha^2V_1+o(\alpha^2), \tag{5.3}$$

$$\bar{M}^2/(1-\bar{M}^2) = N_0(z^*)+\alpha^2N_1(z^*)+o(\alpha^2), \tag{5.4}$$

then it is easy to show that

$$V_0 = \int_0^{z^*}\frac{T(z_1^*)}{N_0(z_1^*)}\mathrm{d}z_1^* \quad \text{for } z^* < z_c^* \tag{5.5a}$$

and

$$V_0 = -\int_{z^*}^{\infty}\frac{T(z_1^*)}{N_0(z_1^*)}\mathrm{d}z_1^*+\tilde{B}_0 \quad \text{for } z^* > z_c^*, \tag{5.5b}$$

where $\tilde{B}_0$ is a constant which is to be determined by the match through the critical layer at $z^* = z_c^*$, where $u(z_c^*) = 1-1/M_\infty$, and with an exponentially decaying outer solution. Also

$$\frac{N_0}{T}\frac{\mathrm{d}V_1}{\mathrm{d}z^*} = \int_0^{z^*}TV_0\bar{M}_0^2\,\mathrm{d}z_1^*-\frac{N_1}{N_0} \quad \text{for } z^* < z_c^*, \tag{5.6a}$$

$$\frac{N_0}{T}\frac{\mathrm{d}V_1}{\mathrm{d}z^*} = \int_{z_c^*}^{z^*}TV_0\bar{M}_0^2\,\mathrm{d}z_1^*-\frac{N_1}{N_0}+\tilde{B}_1 \quad \text{for } z^* > z_c^*, \tag{5.6b}$$

where $$\bar{M}_0^2(z^*) = \lim_{\alpha\rightarrow 0}\bar{M}^2(z^*),$$

as obtained from (1.2) by setting $c = 1-1/M_\infty$, so that $\bar{M}_0^2(z_c^*) = 0$. Again $\tilde{B}_1$ is an arbitrary constant although its value will be immaterial to this study.

We first require the form of V_0, V_1 for $z^* \gg 1$. Since it follows from (5.2), (1.2) and (2.6) that $N_0(z^*) = O(z\exp(\frac{1}{4}z^2))$ for $z^* \gg 1$ we see from (5.5*b*) that $V_0 \to \tilde{B}_0$ as $z^* \to \infty$. Also we find from (1.2) that

$$\frac{N_1}{N_0^2} \approx -2\tilde{c}M_\infty \quad \text{when } z^* \gg 1, \tag{5.7}$$

and hence, from (5.6*b*), that

$$V_1 \approx 2\tilde{c}M_\infty z^*. \tag{5.8}$$

Thus the leading terms of V are

$$V \approx \tilde{B}_0 + 2\tilde{c}M_\infty \alpha^2 z^*, \tag{5.9}$$

when $z^* \gg 1$, and we now evaluate $\tilde{B}_0$ by matching with an outer layer in which the two terms of (5.9) are of the same size. In this outer region (5.1) reduces to

$$\frac{\mathrm{d}^2 V}{\mathrm{d}z^{*2}} - 2\tilde{c}M_\infty \alpha^4 V = 0, \tag{5.10a}$$

so that
$$V = \exp[-(2\tilde{c}M_\infty)^{\frac{1}{2}}\alpha^2 z^*], \tag{5.10b}$$

where the square root with a positive real part is to be chosen. Thus the match between (5.9) and (5.10) implies that

$$\tilde{B}_0 = -(2\tilde{c}M_\infty)^{\frac{1}{2}}, \tag{5.11}$$

so that $\operatorname{Re}\tilde{B}_0 < 0$.

Now at $z^* = z_c^*$ the integrand in (5.5) has a double pole and if we write

$$\frac{\bar{M}_v^2}{T(1-\bar{M}_0^2)} = \tfrac{1}{2}(z^*-z_c^*)^2 Q_c'' + \tfrac{1}{6}(z^*-z_c^*)^3 Q_c''' + \dots, \tag{5.12}$$

we have, from (5.5*a*), that, for $z^* < z_c^*$,

$$V_0 = \int_0^{z^*} \left[\frac{T(z_1^*)}{N_0(z_1^*)} - \frac{2}{(z_1^*-z_c^*)^2 Q_c''} + \frac{4}{3}\frac{z_c^*}{z_1^{*2}-z_c^{*2}}\frac{Q_c'''}{Q_c''^2}\right]\mathrm{d}z_1^* - \frac{2}{Q_c''}\frac{z^*}{z_c^*(z^*-z_c^*)} + \frac{2}{3}\frac{Q_c'''}{Q_c''^2}\log\frac{z_c^*+z^*}{z_c^*-z^*} \tag{5.13a}$$

and from (5.5*b*) that, for $z^* > z_a^*$,

$$V_0 = -\int_{z^*}^{\infty} \left[\frac{T(z_1^*)}{N_0(z_1^*)} - \frac{2}{(z_1^*-z_0^*)^2 Q_c''} + \frac{4}{3}\frac{z_c^*}{z_1^{*2}-z_c^{*2}}\frac{Q_c'''}{Q_c''^2}\right]\mathrm{d}z_1^* - \frac{2}{(z^*-z_c^*)Q_c''} + \frac{2}{3}\frac{Q_c'''}{Q_c''^2}\log\frac{z^*+z_c^*}{z^*-z_c^*} + \tilde{B}_0. \tag{5.13b}$$

We now determine $\tilde{B}_0$ by requiring that (5.13*b*) is the analytic continuation of (5.13*a*) on the assumption that $\operatorname{Im} c > 0$. This implies that

$$\log(z^*-z_c^*) = \log(z_c^*-z^*) - \mathrm{i}\pi, \tag{5.14}$$

since $u'(z_c^*) > 0$. It now follows from (5.13) that

$$\tilde{B}_0 = I_c - \frac{2}{Q_c'' z_c^*} - \frac{2}{3}\frac{Q_c'''}{Q_c''^2}\mathrm{i}\pi, \tag{5.15}$$

where

$$I_c = \int_0^\infty \left[\frac{T(z_1^*)}{N_0(z_1^*)} - \frac{2}{(z_1^*-z_c^*)^2 Q_c''} + \frac{4}{3}\frac{z_c^*}{z_1^{*2}-z_c^{*2}}\frac{Q_c'''}{Q_c''^2}\right] dz_1^*. \tag{5.16}$$

Two requirements must be fulfilled, namely $\operatorname{Im}\tilde{c} > 0$ and $\operatorname{Re}\tilde{B}_0 < 0$. Since $2\tilde{c}M_\infty = \tilde{B}_0^2$, together these imply that it is necessary that

$$\operatorname{Re}\tilde{B}_0 < 0, \quad \operatorname{Im}\tilde{B}_0 < 0. \tag{5.17a, b}$$

If, for any M_∞, either of (5.17) is not satisfied we may infer that the sonic mode that exists with $\alpha = 0$ and $c = 1 - 1/M_\infty$ has no neighbouring unstable modes.

To examine whether (5.17) is satisfied for varying γ and M_∞ it is necessary to evaluate I_c numerically. However if $M_\infty - 1 \ll 1$ or $M_\infty \gg 1$ this may be done analytically. We first evaluate Q_c'', Q_c''' to give

$$Q_c'' = 2M_\infty^2 u_c'^2/T_c^2, \tag{5.18a}$$

$$Q_c''' = \frac{6M_\infty^2 u_c'^2}{T_c^2}\left(\frac{u_c''}{u_c'} - \frac{2T_c'}{T_c}\right). \tag{5.18b}$$

When $0 < M_\infty - 1 \ll 1$, z_c^* is very near the wall with

$$u_c = 1 - \frac{1}{M_\infty} \approx \lambda z_c^* \quad (\lambda = 0.332) \tag{5.19}$$

and the leading terms of (5.15) are not derived from I_c. We find, on using (5.18), that

$$\tilde{B}_0 \approx -\frac{1}{4}\frac{(\gamma+1)^2}{(M_\infty - 1)\lambda} - \frac{(\gamma^2-1)}{\lambda} i\pi (M_\infty - 1) \tag{5.20}$$

gives the leading terms for $M_\infty - 1$ small, and (5.17) is satisfied.

When $M_\infty \gg 1$, z_c^* is right at the edge of the boundary layer and (2.6) may be used. With $z_c = z_c^* - 1.721$ we obtain

$$\frac{\mu}{z_c} e^{-\frac{1}{4}z_c^2} \approx \frac{1}{M_\infty}, \tag{5.21}$$

and that

$$Q_c'' \approx \frac{z_c^2}{2(\gamma-1)^2 M_\infty^2}, \quad Q_c''' \approx \frac{3z_c^3}{4(\gamma-1)^2 M_\infty^2} \tag{5.22}$$

To evaluate I_c in this limit we replace (5.16) by

$$\frac{dI_c^*}{dz^*} = F(z^*), \quad I_c^*(0) = 0, \tag{5.23}$$

where $F(z^*)$ is the integrand of (5.16). Then $I_c = I_c^*(\infty)$. The function $F(z^*)$ has to be considered in three regions, firstly where $z^* = O(1)$, secondly where $z^* \approx z_c^*$, and finally when $z^* \gg 1$ and exponentially small terms can be ignored.

When $z^* = O(1)$, $M_\infty \gg 1$,

$$F(z^*) \approx \tfrac{1}{2}(\gamma-1)M_\infty^2(1+u)\left[\tfrac{1}{2}(\gamma-1)(1+u) - (1-u)\right] \equiv F_0(z^*), \tag{5.24}$$

say, so that in this region

$$I_c^* \approx \int_0^{z^*} F_0(z_1^*)\, dz_1^*. \tag{5.25}$$

When $z^* \approx z_c^*$ as given by (5.21) we write

$$z = z_c + Z/z_c, \tag{5.26}$$

and use (2.6) to replace T, N_0 by their form in this neighbourhood. Here

$$F(z^*) \approx (\gamma-1)^2 M_\infty^2 \left[\frac{e^{-Z}}{(1-e^{-\frac{1}{2}Z})^2} - \frac{4}{Z^2} + \frac{2}{Z}\right], \tag{5.27}$$

so that

$$I_c^* \approx \frac{2(\gamma-1)^2}{z_c} M_\infty^2 \left[\log\frac{Z}{1-e^{-\frac{1}{2}Z}} - \frac{1}{1-e^{-\frac{1}{2}Z}} + \frac{2}{Z} + B_c\right], \tag{5.28}$$

where B_c is a constant to be determined by matching. When $z^* \gg 1$, we obtain, on use of (5.21),

$$F(z^*) \approx \frac{4(\gamma-1)^2 M_\infty^2}{z^{*2} - z_c^{*2}}, \tag{5.29}$$

so that
$$I_c^* \approx \frac{2(\gamma-1)^2}{z_c} M_\infty^2 \left[\log\frac{z-z_c}{z+z_c} + A_c\right]. \tag{5.30}$$

Thus
$$I_c = I_c^*(\infty) = \frac{2(\gamma-1)^2 M_\infty^2}{z_c} A_c, \tag{5.31}$$

and we now determine A_c, B_c by matching (5.25), (5.28), (5.30). We first match $Z \to \infty$ in (5.28) with $z \to z_c$ in (5.30), and from the constant terms obtain

$$A_c - 2\log z_c = B_c - 1. \tag{5.32}$$

Finally we match $Z \to -\infty$ in (5.28) with $z^* \to z_c^*$ in (5.25) to give

$$B_c = \tfrac{1}{2} z_c^2 \tag{5.33}$$

so that (5.32) and (5.31) now lead to

$$I_c \approx (\gamma-1)^2 M_\infty^2 z_c, \tag{5.34}$$

where, as in (5.21), $z_c \approx 2(\log M_\infty)^{\frac{1}{2}}$.

We may now write down the leading-order terms of $\operatorname{Re}\tilde{B}_0$, $\operatorname{Im}\tilde{B}_0$ as given by (5.15). They are

$$\tilde{B}_0 \approx (\gamma-1)^2 M_\infty^2 \left(z_c - \frac{2i\pi}{z_c}\right), \tag{5.35}$$

where the relative error in both terms is $O(z_c^{-1})$. We see immediately that $\operatorname{Re}\tilde{B}_0$ has the wrong sign, and indeed numerical evaluation of $\tilde{B}_0$ for $\gamma = 1.4$ shows that the sign change occurs for $M_\infty = M_{\text{crit}}$ where M_{crit} is approximately equal to 73.

The analysis of this section has shown that the sonic mode no longer exists as $M_\infty \to \infty$, except possibly as an isolated neutral mode at $\alpha = 0$. Two questions remain: first, what is the fate of the near-neutral mode as M_∞ increases through M_{crit} with $0 < \alpha \ll 1$; secondly, what is the limit of the vorticity mode as $\alpha \to 0$? We are unable to answer the second at present except to note that it is not the neutral sonic mode because at large M_∞ we have shown that this has no neighbouring unstable modes. We may, however, make a comment on the first question. From (5.11) we have that

$$2\tilde{c}M_\infty = \tilde{B}_{0r}^2 - \tilde{B}_{0i}^2 + 2i\tilde{B}_{0r}\tilde{B}_{0i} \tag{5.36}$$

where $\tilde{B}_{0r}$ was required to be negative so that (5.10*b*) decays, and $\tilde{B}_{0i}$ must correspondingly be negative so that $\mathrm{Im}\,\tilde{c} > 0$ as assumed in the derivation of (5.14). One would suspect that the unstable mode that exists for small values of α when $M_\infty < M_{crit}$ becomes unstable at a non-zero value of α when $M_\infty > M_{crit}$ although to confirm this it would be necessary to compute the term $o(\alpha^2)$ in (5.3), (5.4) – a formidable task. At $M_\infty = M_{crit}$ the limiting solution represents an outgoing wave rather than an exponentially decaying mode. For such a wave it is necessary that $\tilde{c}$ in (5.10) is real and negative and certainly, when $\gamma = 1.4$ at least as has been verified by the numerical work, $\tilde{c}$ as given by (5.36) is real and negative when $M_\infty = M_{crit}$. It thus appears possible that an outgoing neutral wave bifurcates from the sonic mode at $M_\infty = M_{crit}$, and for $M_\infty > M_{crit}$ unstable waves will not attain $\alpha = 0$, but will become neutral as outgoing waves for non-zero values of α.

6. Discussion

The neutral modes of inviscid instability of a flat-plate compressible boundary layer are essentially of two types, namely acoustic and vorticity modes. When the modes are non-inflectional, i.e. there is no generalized inflection point or critical layer, then the wavespeed c is unity and the wavenumber α is, for each mode, a continuous monotonic decreasing function of M_∞, and the perturbations are all acoustic modes. However, for the inflectional modes, for which $c = c_s$, a vorticity mode may be identified in that each mode contributes a portion of curve with α an increasing function of M_∞. At any M_∞ there is just one such vorticity mode. In the limit $M_\infty \to \infty$ these portions may be regarded as continuous, and this mode and neighbouring unstable modes have been analysed here in §3. In this high-Mach-number limit the remaining portions of the inflectional neutral modes become, to leading order, indistinguishable from the non-inflectional neutral modes, with α a continuous function of M_∞. Thus, for $M_\infty \gg 1$, the acoustic modes are cut, in the $M_\infty\,\alpha$-plane, by the vorticity mode, although at finite M_∞ there is no such intersection but merely an interchange of mode identity. The structure of this near-linking is given in §4 where it is shown that the separation distance is exponentially small when M_∞ is large. The vorticity mode is important because the neighbouring unstable modes have, when $M_\infty \gg 1$, higher growth rates than do those neighbouring the neutral acoustic modes, although at moderate values of M_∞ it is an acoustic mode, the second mode, that is the most unstable. The form of this vorticity mode for very small values of α is an open question.

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